THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: Yvan Leblanc

TITLE OF THESIS: On Extended Objects and their Quantum Numbers in Quantum Field Theory

DEGREE FOR WHICH THESIS WAS PRESENTED: M. Sc.

YEAR THIS DEGREE GRANTED: 1982

THE UNIVERSITY OF ALBERTA

ON EXTENDED OBJECTS AND THEIR QUANTUM NUMBERS IN QUANTUM FIELD THEORY

by

YVAN LEBLANC

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF PHYSICS

EDMONTON, ALBERTA

FALL, 1982

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ON EXTENDED OBJECTS AND THEIR QUANTUM NUMBERS IN QUANTUM FIELD THEORY submitted by Yvan Leblanc in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

The Quantum Field Theory of a self-interacting scalar field is reviewed in the case where an extended object is present. As a consequence of a new type of interaction, the soliton-quanta interaction, the physical observables differ substantially from the case without an extended object. A fundamental result is the appearance of the quantum coordinate which emerges as a zero-energy Boson mode taking care of the rearrangement of the translational symmetry. The latter symmetry was spontaneously broken by the creation of the extended object through the Boson transformation. The existence of the object also alters the structure of the Poincaré generators through the asymptotic and c-Q transmutation conditions as well as the Fock space of the theory which must be enlarged to contain quantum soliton states. The knowledge of the Poincaré generators gives exact information on the dynamical map of the Boson field and leads to the existence of the so-called generalized coordinates.

The review is extended to the Quantum Field Theory of interacting scalar and spinor fields in the presence of the object in 1+1 dimensions. Again we are led to a very rich physical situation where a Fermion zero-energy mode appears beside the quantum coordinate. This Fermion mode is localized on the soliton which then acquires a set of new quantum numbers related to the symmetries of the spinor

field. The structure of the latter field is then determined from its properties under the Poincaré transformations and is used to show the existence of a hidden supersymmetry of the theory at the level of the physical fields.

Finally, the above considerations are shown to be of relevance in Condensed Matter Physics where solitons carrying fractional quantum numbers have been observed.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

INTRODUCTION

The purpose of the present work is to review the Quantum Field Theory of a self-interacting scalar field and the model describing interacting scalar and spinor fields in the case where an extended classical object interacts with the quanta.

The first chapter will outline a formal method of introducing extended objects in Quantum Field Theory, the Boson Transformation, in the context of a self-interacting scalar field. The emergence of a new physical operator, the quantum coordinate, will be shown. The quantum coordinate is a Goldstone mode which appears as a rearrangement of the translational symmetry. It possesses a canonically conjugate momentum which is, through the asymptotic condition, the total momentum of the system. Asymptotic forms for the Poincaré generators will be given and the c-Q transmutation condition will argue for the existence of the generalized coordinates. These are operator expressions determining in which combination the c-number coordinates appear with the physical operators of the theory in the dynamical map of the Boson transformed Heisenberg field.

The second chapter will review the theory of interacting scalar and spinor fields with a topological soliton in 1+1 dimensions. Very interesting phenomena occur in this theory such as the emergence of a Fermion zero-energy mode, a hidden supersymmetry at the level of the physical

fields and the experimentally verifiable charge fractionalization mechanism. The quantum numbers in general carried by the soliton will be obtained through a calculation scheme to which the author has contributed. The charge fractionalization appears in this calculation. Because of the experimentally verifiable aspects of the latter mechanism in Condensed Matter Physics, a model of the polyacetylene molecule as well as of the niobium triselenide molecule will be qualitatively described.

It is very interesting to note that in Quantum Field Theory with extended objects, the nature of the fields can be three-fold. They can show classical, quantum mechanical and quantum field theoretic modes of behaviour. This immediately points out to us that the quantum-classical duality is not linear as one is used to think. The classical limit is usually thought of as the limit as Planck's constant goes to zero. However, as will be shown in this work, classical behaviour may also arise when we deal with many bodies. The extended object, created by the local condensation of many physical quanta in the vacuum will show quantum or classical behaviour according to whether or not the expectation value of the quantum fluctuations is large compared to the classical background. This non-linearity of the quantum-classical duality is well illustrated by the previously mentioned

charge fractionalization mechanism. The fractional electric charge, which is of classical nature (macroscopic), emerges from the quantum (microscopic) properties of the object. On the other hand, when we deal with gauge theories with extended objects, the so-called flux quantization is a phenomenon of purely classical nature which leads to quantum states (topological quantum numbers) of macroscopic origin. It is easy to recognize that the non-linearity of the classical limit is partly due to the existence of a new type of interaction, the soliton-quanta interaction.

In this work c (speed of light) is set equal to 1.

I. THE BOSON TRANSFORMATION APPROACH

A Summary of Previous Approaches

Extended objects in Quantum Field Theory constitute a very active topic of research today. Since this subject is very extensive, the purpose of this section is not to discuss at length the many different existing approaches but rather to outline them in an introductory and incomplete way. For more details the reader is referred to the review articles of Rajaraman[41], Coleman[6], Gervais and Neveu[13], Jackiw[20], Faddeev and Korepin[9] and more recently Umezawa and Matsumoto[52].

The introduction of extended objects as classical objects in Quantum Field Theory follows different routes according to the problem one wants to solve. For example, in Hadron Physics one introduces bags or strings in phenomenological models as fundamental objects essentially distinct from the quantum fields. In the bag model of hadrons, the bag plays a dynamical role in the confinement of quarks. These quarks are quantum fields interacting with gluons inside the bag.

In Solid State Physics one can describe dislocations in crystals as classically behaving objects interacting with quantum fields such as phonons. Vortices in superconductors, magnetic domains in ferromagnets as well as boundary domain emerging from the symmetry breaking of the chain structure of a polyacetylene molecule can be described as

classical extended objects with relevant properties. Unlike the case of phenomenological bag or string models in Hadron Physics, the latter objects are of quantum origin. Physically however, it may well be that bags and strings are of quantum origin. It is known that this is indeed the case for the Nielsen-Olesen string.

A very common method of introducing these classical objects of quantum origin in local Quantum Field Theory consists of finding the classical solutions of the field equations which are finitely extended in space and which are non-dissipative. These solutions are usually called solitons for 1+1 dimensional models. This classical field is then associated with the vacuum expectation value of the quantum field and is therefore intimately related to the phenomenon of spontaneous symmetry breaking.

Once stable classical solutions have been found, quantum corrections can then be calculated by various methods.

Semiclassical methods developed by Dashen, Hasslacher and Neveu[7,8] and by Korepin and Faddeev[25] are based on the Feynman path integral quantization formalism and make use of the Wentzel-Kramers-Brillouin (WKB) approximation to calculate quantum effects.

Another approach to quantization of classical solutions, developed by Goldstone and Jackiw[16], starts with a fully quantized field theory which is expanded in a Born-Oppenheimer fashion. The first term in the

expansion is the classical solution of the field equations. In this approach, a richer Hilbert space is postulated and later checked for self-consistency. Physical particle states appear along with quantum soliton states. These are eigenstates of the total momentum and energy. The sector of the Hilbert space containing the soliton states together with the particle states is called the soliton sector. In this approach however, when one wants to quantize about c-number static fields, one does not know about which solution to expand since a spatial translation maps one solution into another. The introduction of a new quantum operator called the collective coordinate allows us to restore translational symmetry and do a systematic perturbation expansion about static classical solutions. The collective coordinate method for the soliton problem has been introduced through the Feynman path integral formalism by Gervais and Sakita[14], Callan and Gross[3] and also Korepin and Faddeev[25]. It has been developed using the canonical formalism by Christ and Lee[5] and Tomboulis[51].

Let us note now that it is possible to find non-singular non-dissipative stable c-number solutions to the field equations using topological arguments[6,52]. There are usually non-trivial conservation laws (topological conservation laws) associated with these classical solutions. These conservation laws however are not related to the symmetries of the theory. They tell us that the space of extended object-type of solutions of the field equations is

not simply connected. This means that one can classify solutions into homotopy classes where two solutions belonging to the same class are continuously deformable into each other. Solutions belonging to different classes cannot decay into one another and are therefore stable with respect to each other unless a soliton meets its corresponding antisoliton. Each class is characterized by an integer called the winding number or topological quantum number. Though topological conservation laws do not depend on the symmetries of the theory, which topologically conserved number is quantized (i.e. what kind of topological quantum number) is controlled by symmetries. The existence of topological quantum numbers insures that the stability criterion is satisfied. When one matches solutions belonging to different classes, a discontinuity exists in the matching region and the resulting extended object is said to be topological since it is not single-valued in some definite region of space-time. The discontinuity is called a topological singularity. Extended objects with topological singularities are often the most interesting ones because of their stability and because they naturally introduce a gauge and lead to the so-called flux quantization, monopole charge quantization, etc. When we combine this concept of gauge with a gauge field, we are led to the particularly interesting gauge theories.

Finally, objects carrying other types of singularities (essential singularities) are irrelevant to Physics.

The Boson Transformation and the Quantum Coordinate

The Boson Transformation method[29,30,38,54] constitutes a formal way of introducing extended objects in Quantum Field Theory. In this method, the extended object is created as a result of the condensation of physical Boson fields in the vacuum. This condensation can be expressed as a local c-number physical Boson field translation leading to a space-time dependent vacuum expectation value of the physical field. The condensation of many physical fields lead to a space-time dependent vacuum expectation value (i.e. the order parameter) of the Heisenberg field through its dynamical map which is expressed as linear combinations of normal products of physical fields. This space-time dependent order parameter is then identified with the classical extended object. The creation of the object is therefore based on the duality between Heisenberg and physical field operators.

Let us perform the Boson Transformation on a free (physical) scalar field operator $\phi(x)$,

$$\phi(x) \to \phi^f(x) = \phi(x) + f(x) \quad , \tag{1.1}$$

where $f(x)$ is a c-number function satisfying the same free field equation as $\phi(x)$,

$$\Lambda(\partial)\phi(x) = \Lambda(\partial)f(x) = 0 \quad . \tag{1.2}$$

Here $\Lambda(\partial)$ is an appropriate derivative operator.

We note that when the Boson function $f(x)$ has a Fourier transform, the Boson Transformation is equivalent to a translation of the annihilation operator. The vacuum is then an eigenstate of the annihilation operator implying that it is a coherent state[15].

Introduce the following Heisenberg equation for the Heisenberg field $\psi(x)$,

$$\Lambda(\partial)\psi(x) = F[\psi(x)] \tag{1.3}$$

where $F[\psi]$ is a functional of $\psi(x)$ which is itself a functional of the physical field $\phi(x)$ through its dynamical map. One can then write the following Yang-Feldman equation,

$$\psi(x) = \phi(x) + \Lambda^{-1}(\partial)F[\psi(x)] \ . \tag{1.4}$$

Here $\psi(x)$ is the renormalized Heisenberg field and $F[\psi]$ contains the renormalization counterterms as well as the wave function renormalization factor. Writing the Yang-Feldman equation as the dynamical map,

$$\psi(x) = \psi[x;\phi] \tag{1.5}$$

which is a linear combination of normal products of the physical fields, we will obtain the following dynamical map for the Heisenberg field after the Boson Transformation

$$\psi(x) \to \psi^f(x) = \psi[x;\phi+f] \ . \tag{1.6}$$

The Boson Transformation theorem is:

Both Heisenberg field operators $\psi(x)$ and $\psi^f(x)$ satisfy the same Heisenberg equation.

A simple proof follows from the fact that the Yang-Feldman equation (1.4) holds true when $\phi(x)$ is replaced by $\phi(x)+f(x)$. The proof of this theorem without the use of the Yang-Feldman equation uses the fact that the Boson Transformation commutes with any derivative operation as well as with the product of operators.

Expanding the dynamical map of $\psi(x)$ as,

$$\psi(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4z_1 \ldots d^4z_n \, c(x; z_1 \ldots z_n) :\phi(z_1) \ldots \phi(z_n): \tag{1.7}$$

one then obtains the dynamical map of $\psi^f(x)$ by re-summing as follows,

$$\psi^f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} :\psi_f^{(n)}(x): \tag{1.8a}$$

where

$$\psi_f^{(n)}(x) = \int d^4z_1 \ldots d^4z_n \, c_f(x; z_1 \ldots z_n) \phi(z_1) \ldots \phi(z_n) \tag{1.8b}$$

and

$$c_f(x; z_1 \ldots z_n) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \int d^4y_1 \ldots d^4y_\ell \, c(x; z_1 \ldots z_n, y_1 \ldots y_\ell) \times f(y_1) \ldots f(y_\ell) . \tag{1.8c}$$

The leading orders of the latter expansion are seen to be,

$$\psi^f(x) = c_f(x) + \int d^4z \, c_f(x;z) \phi(z) + \ldots . \tag{1.9}$$

The vacuum expectation value of $\psi^f(x)$ is then,

$$\langle 0 | \psi^f(x) | 0 \rangle \equiv \Phi^f(x) = c_f(x) . \tag{1.10}$$

This space-time dependent order parameter will be seen to obey an Euler-type field equation and will describe the dynamics of the extended object of our theory. Furthermore the linear term in $\phi(x)$ in the dynamical map (1.9) is already seen to be altered by the presence of the extended object through the c-number coefficient $c_f(x;z)$. This tells us that the physical particle representation of our Hilbert space will be altered by the extended object and allows us to foresee the eventual emergence of quantum soliton states.

Equation (1.9) suggests that $\psi^f(x)$ be expanded in a functional Taylor series,

$$\psi^f(x) = \Phi^f(x) + \int d^4z\ \phi(z) \frac{\delta}{\delta f(z)}\ \Phi^f(x)$$
$$+ \frac{1}{2}\int d^4z_1\ d^4z_2 : \phi(z_1)\phi(z_2) : \frac{\delta^2}{\delta f(z_1)\delta f(z_2)}\ \Phi^f(x)$$
$$+ \ldots \quad . \tag{1.11}$$

A comparison with (1.8) leads to,

$$c_f(x;z_1 \ldots z_n) = \frac{\delta^n}{\delta f(z_1)\ldots\delta f(z_n)}\ \Phi^f(x)\ . \tag{1.12}$$

Defining the operator,

$$\delta_f \equiv \int d^4z\ \phi(z)\ \frac{\delta}{\delta f(z)}\quad , \tag{1.13}$$

one gets the following simple form for $\psi^f(x)$,

$$\psi^f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} : (\delta_f)^n \Phi^f(x) : \quad . \tag{1.14}$$

Therefore,

$$\psi_f^{(n)}(x) = (\delta_f)^n \phi^f(x) \tag{1.15}$$

and

$$\psi_f^{(n+1)}(x) = \delta_f \, \psi_f^{(n)}(x) \quad . \tag{1.16}$$

Equations (1.15) and (1.16) show that one can solve completely, at least formally, the dynamical map for $\psi^f(x)$ to all orders once its vacuum expectation value (the classical extended object) has been calculated. In order to obtain this quantity however, one must use an approximation scheme, since we are confronted with a highly non-linear differential equation. It has been shown by Matsumoto et al.[35] that the vacuum expectation value for $\psi^f(x)$ satisfies the field equation (1.3) in the tree approximation. This is so since

$$<0|F[\psi^f]|0> = F[<0|\psi^f|0>] + \text{loop diagrams} \ . \tag{1.17}$$

Identifying with a hat the set of c-numbers or Heisenberg field operators in the tree approximation, we write (1.3) as

$$\Lambda(\partial)\hat{\psi}^f = F[\hat{\psi}^f] \tag{1.18}$$

which can be re-written as,

$$\Lambda(\partial)\left[\sum_{n=0}^{\infty} \frac{1}{n!} :\hat{\psi}_f^{(n)}:\right] = F\left[\sum_{n=0}^{\infty} \frac{1}{n!} :\hat{\psi}_f^{(n)}:\right] \ . \tag{1.19}$$

Expanding the right-hand side as a functional Taylor series about the vacuum expectation value of $\psi^f(x)$ in the tree approximation (denoted $\hat{\Phi}^f(x)$) and making use of the polynomial expansion, disregarding the non-commutativity

among different orders $\hat{\psi}_f^{(n)}(x)$ because of the tree approximation, one then obtains the following set of differential equations,

$$\Lambda(\partial)\hat{\psi}_f^{(n)}(x) = n! \sum_k F^{(k)}[\hat{\Phi}^f(x) \prod_{i=1}^{\infty} \frac{1}{k_i!}\left(\frac{\hat{\psi}_f^{(n_i)}(x)}{n_i!}\right)^{k_i} \tag{1.20a}$$

restricted by,

$$\sum k_i = k \quad ; \quad \sum k_i n_i = n \qquad \text{for} \quad 0 < n_i < n \, . \tag{1.20b}$$

Here,

$$\frac{\delta^k}{\delta\Phi(x_1)\ldots\delta\Phi(x_k)} F[\Phi(x)] \equiv F^{(k)}[\Phi(x)]\delta^{(4)}(x-x_1)\ldots\delta^{(4)}(x-x_k)\, . \tag{1.21}$$

For $n = 0$ and $n = 1$, we obtain respectively ($\hat{\psi}_f^{(o)} \equiv \hat{\Phi}^f$),

$$\Lambda(\partial)\hat{\Phi}^f(x) = F[\hat{\Phi}^f(x)] \tag{1.22}$$

and

$$\{\Lambda(\partial) - F^{(1)}[\hat{\Phi}^f(x)]\}\hat{\psi}_f^{(1)}(x) = 0 \, . \tag{1.23}$$

Equation (1.22) is simply the Euler equation for the extended object while equation (1.23) represents a Schrödinger-type equation with c-number potential $F^{(1)}[\hat{\Phi}^f(x)]$. It is called the self-consistent potential by Matsumoto et al.[27] and equation (1.23) is called the stability equation by Jackiw[20]. It describes a quantum field interacting with the extended object. Depending upon the form for the self-consistent potential, $\hat{\psi}_f^{(1)}$ will develop scattered and/or bound state modes. Since it is the linear term in the dynamical map it will be chosen to realize an irreducible representation of the Hilbert space

of the theory. Note that in the latter formalism, it is implicitly assumed that there is no composite particle. When this is the case, we know that a new physical field operator corresponding to the composite particle must be introduced in the existing set of physical field operators for the sake of completeness. This bosonic bound state can then condense in the vacuum like any other physical Boson fields. In the Nambu model, for instance, the Goldstone Boson is composite and once it is Boson transformed, one is led to extended objects of vortex-type[30,31].

An important feature which arises from the above considerations is the emergence of a new quantum mechanical operator called the quantum coordinate. It is an operator similar to the collective coordinate mentioned in the previous section. Although they play the same role, they are not quite identical. It is known that the quantum coordinate commutes with the annihilation and creation operators while the collective coordinate does not.

To see how the quantum coordinate emerges from the theory, let us discuss the situation in which the extended object is time independent in its proper frame. One then separates $\hat{\psi}_f^{(1)}(x)$ into scattered solutions and bound state solutions with the respective wave functions satisfying the stability equations

$$\{\Lambda(\vec{\partial},\omega_k) - F^{(1)}[\hat{\phi}^f(\vec{x})]\}u(\vec{x},\vec{k}) = 0 \tag{1.24a}$$

and

$$\{\Lambda(\vec{\partial},\omega_i) - F^{(1)}[\hat{\Phi}^f(\vec{x})]\}u_i(\vec{x}) = 0 \qquad (1.24b)$$

where

$$\Lambda(\vec{\partial},\omega) \equiv \Lambda(\partial)\Big|_{\partial_t=-i\omega} \quad . \qquad (1.24c)$$

One can choose the above wave functions to satisfy the following orthogonality and completeness conditions,

$$\int d^3x \ u^*(\vec{x},\vec{k})u(\vec{x},\vec{\ell}) = \delta(\vec{k}-\vec{\ell}) \quad , \qquad (1.25a)$$

$$\int d^3x \ u_i^*(\vec{x}) \ u_j(\vec{x}) = \delta_{ij} \quad , \qquad (1.25b)$$

$$\int d^3x \ u_i^*(\vec{x}) \ u(\vec{x},\vec{k}) = 0 \qquad (1.25c)$$

and

$$\int d^3k \ u(\vec{x},\vec{k})u^*(\vec{y},\vec{k}) + \sum_i u_i^*(\vec{y})u_i(\vec{x}) = \delta(\vec{x}-\vec{y}) \quad . \qquad (1.26)$$

It is now possible to show that under spatial translation, the corresponding variation of the extended classical field satisfies the stability equation (1.24b) for bound states with zero energy. For a non-spherically symmetric extended object, the variation of the classical field under spatial rotations also satisfies equation (1.24b) with zero-energy solutions. The problem of non-spherically symmetric extended objects has been treated recently by Papastamatiou et al.[39] in 2+1 dimensions. A new angular operator will emerge from rotational invariance while the quantum coordinate will emerge from translational invariance. Here, we will restrict ourselves to spherically symmetric extended objects. Therefore only the quantum coordinate will appear.

Taking the spatial gradient of the Euler equation (1.22) leads to,

$$\Lambda(\vec{\partial},0)\nabla\hat{\Phi}^{f}(\vec{x}) = F^{(1)}[\hat{\Phi}^{f}(\vec{x})]\nabla\hat{\Phi}^{f}(\vec{x}) \quad . \tag{1.27}$$

It is then obvious that $\nabla\hat{\Phi}^{f}(\vec{x})$ is a solution of (1.24b) with zero energy. To each spatial direction is associated a zero-energy solution $\partial_i\hat{\Phi}^{f}(\vec{x})$ $(i = 1,2,3)$.

We now turn to the problem of constructing the quantum fields which describe the theory including the three translational modes. Fields with wave functions satisfying (1.24a) and (1.24b) with non-zero energy can be written respectively as,

$$\hat{\chi}_s(x) \equiv \int d^3k \; \frac{\hbar^{\frac{1}{2}}}{\sqrt{2\omega_k}} \; [u(\vec{x},\vec{k})\alpha(\vec{k})e^{-i\omega_k t} + \text{h.c.}] \tag{1.28}$$

and

$$\hat{\chi}_b(x) \equiv \sum_{i>3} \frac{\hbar^{\frac{1}{2}}}{\sqrt{2\omega_i}} \; [u_i(\vec{x}) \; \alpha_i \, e^{-i\omega_i t} + \text{h.c.}] \tag{2.29}$$

where $\alpha(\vec{k})$ and α_i are scattered and bound state annihilation operators and h.c. means hermitian conjugate.

Since the translational modes are zero-energy modes, we write them as,

$$\text{T.M.} \equiv \sum_{i=1}^{3} u_i(\vec{x}) \; \beta_i \tag{1.30}$$

where β_i is a new quantum mechanical operator. It will be shown to possess a canonical conjugate if $\hat{\psi}^{f}(x)$ is to obey an equal time commutation relation (E.T.C.R.). Normalizing the translational mode using (1.25b), one then writes,

$$\text{T.M.} = -\vec{q} \cdot \nabla\hat{\Phi}^{f}(\vec{x}) \quad . \tag{1.31}$$

Here $\vec{q}$ is the quantum coordinate. It differs from $\vec{\beta}$ defined previously by a normalization. One can now write $\hat{\psi}_f^{(1)}(x)$ as,

$$\hat{\psi}_f^{(1)}(x) = \hat{\chi}_s(x) + \hat{\chi}_b(x) - \vec{q} \cdot \nabla\hat{\Phi}^{f}(\vec{x}) \quad . \tag{1.32}$$

Now we assume the following E.T.C.R. for the Boson transformed Heisenberg field $\hat{\psi}^f(x)$,

$$[\hat{\psi}^f(\vec{x},t_x),\dot{\hat{\psi}}^f(\vec{y},t_y)]\delta(t_x-t_y) = i\hbar\delta^{(4)}(x-y) \quad . \tag{1.33}$$

This leads to

$$[\hat{\psi}_f^{(1)}(\vec{x},t_x),\dot{\hat{\psi}}_f^{(1)}(\vec{y},t_y)]\delta(t_x-t_y) = i\hbar\delta^{(4)}(x-y) \tag{1.34}$$

in the tree approximation.

Since $\vec{q}$ commutes with the annihilation and creation operators of the theory, equation (1.34) implies that there exists a momentum operator $\vec{p}$ proportional to $\dot{\vec{q}}$ which satisfies the following commutation relation with the quantum coordinate $\vec{q}$,

$$[q_i,p_j] = i\hbar\delta_{ij} \quad . \tag{1.35}$$

Since $\vec{q}$ and $\vec{p}$ are canonically conjugate quantum mechanical operators, they must satisfy the Heisenberg inequalities. Furthermore, from (1.32) we see that the no-particle state of the theory is a pure quantum mechanical representation for $\vec{p}$ and $\vec{q}$. It is the state of the quantum soliton and the sector of the Hilbert space which contains the quantum soliton is called the soliton sector.

As for any Heisenberg operator, the time translation is generated by the Hamiltonian. Therefore one has,

$$\vec{Q}(t) \equiv e^{iHt}\,\vec{q}(0)\,e^{-iHt} \quad . \tag{1.36}$$

$\vec{Q}(t)$ differs from $\vec{q}(t)$ since it contains annihilation and creation operators through the Hamiltonian. However one can show that there is no difference between the two operators in the tree approximation. One then replaces $\vec{q}(t)$ by $\vec{Q}(t)$ in previous expressions.

Now since $-\vec{Q}(t)\cdot\nabla\hat{\Phi}^f(\vec{x})$ must satisfy (1.23) and since $\nabla\hat{\Phi}^f(\vec{x})$ satisfies equation (1.27), we are led to,

$$\ddot{\vec{Q}} = 0 \tag{1.37}$$

if we assume that the derivative operator $\Lambda(\partial)$ is of second order with respect to the time derivative. This implies that,

$$\vec{Q}(t) = \vec{q} + \dot{\vec{Q}}\,t \tag{1.38a}$$

where

$$\vec{q} \equiv \vec{q}(0) \quad . \tag{1.38b}$$

An important aspect of the quantum coordinate is that it always appears in the combination $\vec{x} - \vec{Q}(t)$ in the dynamical map for $\hat{\psi}^f(x)$. To show it rigorously would require explicit higher order solutions of equation (1.20a). A rough way to perceive the solution is to remark that the following expansion,

$$\hat{\psi}^f(x) = \hat{\Phi}^f(\vec{x}) + \hat{\chi}(x) - \vec{Q}(t)\cdot\nabla\hat{\Phi}^f(\vec{x}) + \ldots \tag{1.39a}$$

where

$$\hat{\chi}(x) \equiv \hat{\chi}_s(x) + \hat{\chi}_b(x) \qquad (1.39b)$$

is indeed a Taylor series for $\hat{\psi}^f(x)$,

$$\hat{\psi}^f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (\vec{Q}\cdot\nabla)^m \left(\sum_{n=0}^{\infty} :\frac{\hat{\chi}^{(n)}(x)}{n!}: \right), \qquad (1.40)$$

where $\hat{\chi}^{(n)}(x)$ is $\hat{\psi}^{(n)}_f(x)$ in which the quantum coordinate is absent. $\hat{\chi}^{(n)}(x)$ obeys equation (1.20a) with $\hat{\chi}^{(1)}(x) \equiv \hat{\chi}(x)$ and $\hat{\chi}^{(o)}(x) \equiv \hat{\Phi}^f(\vec{x})$. Therefore equation (1.40) leads to,

$$\hat{\psi}^f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} : \hat{\chi}^{(n)}(\vec{x}-\vec{Q}(t),t) : \quad , \qquad (1.41)$$

which shows that $\vec{Q}(t)$ always appears in the combination $\vec{x}-\vec{Q}(t)$.

As a final remark in this section, let me add that the Hilbert space of this theory is a direct product of the quantum mechanical realization of the operators $(\vec{q},\vec{p})$ and the Fock space realization of the physical particle annihilation and creation operators. This Hilbert space has been called the extended Fock space and its vacuum can be denoted by,

$$|0\rangle \equiv |0_F\rangle \otimes |\Psi_{p,q}\rangle \quad , \qquad (1.42)$$

where $|0_F\rangle$ is the vacuum of the Fock space and $|\Psi_{p,q}\rangle$ is the pure quantum mechanical soliton state.

When the vacuum in the soliton sector is in the momentum representation, an arbitrary state can be written as,

$$|\psi\rangle = \int d^3\ell \int d^3k_1 \ldots d^3k_n \, g_o(\vec{\ell})\, g_1(\vec{k}_1) \ldots g_n(\vec{k}_n)\, \alpha^\dagger_{k_1} \ldots \alpha^\dagger_{k_n} |\vec{\ell}\rangle \, . \qquad (1.43)$$

The state $|\vec{\ell}\rangle$ is the pure soliton state and is an eigenstate of the operator $\vec{p}$. The set $\{g_i(\vec{k}_i)\}$ in (1.43) is a certain orthonormalized complete set of square-integrable functions. Let us remark that,

$$\alpha_k|\vec{\ell}\rangle = 0 \ . \tag{1.44}$$

Finally the pure soliton state can be written as,

$$|g_o\rangle \equiv \int d^3\ell \; g_o(\vec{\ell})\,|\vec{\ell}\rangle \quad . \tag{1.45}$$

The Asymptotic and c-Q Transmutation Conditions

We have seen in the previous section that the quantum coordinate $\vec{Q}(t)$ always appears in the dynamical map of the Boson transformed Heisenberg field $\hat{\psi}^f(x)$ in the combination $\vec{x}-\vec{Q}(t)$. Furthermore it has been shown how this new Heisenberg operator emerges from translation invariance of the theory.

In Quantum Field Theory it is known that in c-number transformations such as space-time translations, Lorentz transformations as well as spatial rotations are induced through operator (Q-number) transformations. They are called c-Q transmutations. In the case without an extended object, the generators of such transformations are constructed from the physical particle creation and annihilation operators. For example, the asymptotic forms for the time translation generator (asymptotic Hamiltonian) and space translation generator (total momentum) are respectively,

$$H_o = \hbar \int d^3k \; \omega_k \; \alpha_k^\dagger \alpha_k \tag{1.46}$$

and

$$\vec{P} = \hbar \int d^3k \; \vec{k} \; \alpha_k^\dagger \; \alpha_k \quad . \tag{1.47}$$

These operators are the ones corresponding to a free field theory and are weakly equal to the energy and momentum operators calculated from the Lagrangian formalism in the interacting case.

Defining,

$$\alpha_k(\vec{x},t) \equiv \alpha_k \exp -i\hbar[\omega_k t - \vec{k}.\vec{x}] \quad , \tag{1.48}$$

the c-Q transmutation with respect to the time translation can be written as,

$$e^{iH_o a} \alpha_k(\vec{x},t) \, e^{-iH_o a} \equiv \alpha_k'(\vec{x},t) = \alpha_k(\vec{x},t') \tag{1.49}$$

where,

$$t' = t + a \; . \tag{1.50}$$

Similarly we have for spatial translations,

$$e^{i\vec{P}\cdot\vec{a}} \alpha_k(\vec{x},t) \, e^{-i\vec{P}\cdot\vec{a}} \equiv \alpha_k'(\vec{x},t) = \alpha_k(\vec{x}',t) \tag{1.51}$$

where,

$$\vec{x}' = \vec{x} + \vec{a} \; . \tag{1.52}$$

When there is an extended object however, the situation is different. Since we know that the c-number coordinate $\vec{x}$ always appears in the combination $\vec{x}-\vec{q}$ with the quantum coordinate, we obtain the important result that c-number space translations of $\vec{x}$ will be

induced by a c-number translation of the quantum c-ordinate. This already tells us that the annihilation and creation operators need not change under spatial translation and that consequently they commute with the corresponding generator. In turn, this implies that the total momentum is independent of the annihilation and creation operators, which is a drastically different situation from the one without an extended object. The physical meaning of this remarkable result is that since the position of the physical particles is described relative to the position of the object, a spatial translation of these particles is equivalent to a spatial translation of the extended object.

Let us now turn to the asymptotic condition in field theory with an extended object.

In the absence of the object, the usual asymptotic condition in the Lehmann-Symanzik-Zimmermann (LSZ) formalism picks up the terms which behave as free fields in the $t \to \pm\infty$ limit. In the case with an extended object, however, we know that even when this time limit has been performed, the quanta of the theory still interact with the object through the self-consistent potential. A further limiting process is required and in order to obtain truly free behaviour, the limit must spatially separate quanta from the object. Denoting the position of the quantum extended object by $\vec{x}_o + \vec{q}$, one constructs the following asymptotic condition[28],

$$<g_o|\,|\vec{x} - (\vec{x}_o + \vec{q})|\,|g_o> \rightarrow \infty \ . \tag{1.53}$$

The absolute value inside the bra-ket means the root mean square value of $\vec{x} - (\vec{x}_o + \vec{q})$.

Since the dynamical map of $\psi^f(x)$,

$$\psi^f(x) = \psi^f(\vec{x},t;\ \vec{q},\vec{p};\ \alpha^\dagger,\alpha) \tag{1.54}$$

is a weak relation, the asymptotic condition (1.53) implies that $\vec{x}$ and $\vec{q}$ always appear in the combination $\vec{x} - \vec{q}$ in the dynamical map (1.54). This result is exact and is therefore not based on the tree approximation. Now since the c-number spatial translation $\vec{x} + \vec{a}$ can be induced by the Q-number translation $\vec{q} - \vec{a}$ and since the total momentum $\vec{P}$ of the system (calculated from the Lagrangian formalism) generates spatial translations, it follows that,

$$[q_i, P_j] = i\hbar\delta_{ij} \ . \tag{1.55}$$

Therefore,

$$\vec{P} = \vec{p} \equiv <0_F|\vec{P}|0_F> \ . \tag{1.56}$$

Equation (1.56) is a weak relation and tells us that the total momentum of the system can be determined from the momentum of the quantum extended object. This latter equation is of the greatest importance since it tells us that, as stated earlier, the total momentum commutes with the creation and annihilation operators of the theory and therefore that relation (1.47) no longer holds when there is an extended object. Relation (1.56) implies that the total momentum of the system is attached to the soliton.

It is now possible to obtain asymptotic forms for a whole set of relevant operators in Quantum Field Theory such as the generators of the Poincaré transformations. Once these operators are determined it will be possible to construct the dynamical map of $\psi^f(x)$ by an extensive use of the c-Q transmutation condition. This will give exact results independent of any approximation.

Before going on, let us remark that everything so far has been derived under the assumption that the extended object is spherically symmetric and therefore that the quantum coordinate is the only zero-energy mode present in the theory.

From now, we will restrict the theory to the $1+1$ dimensional case. Under such a constraint, there is neither an angular momentum nor a spin operator in the theory. Also a subtle complication has then been avoided. It is related to the c-Q transmutation condition for Lorentz transformations. The phenomenon of Wigner rotation[55,56] would then appear in more spatial dimensions. The Wigner rotation is the rotation involved when a moving object is boosted. Here the quantum extended object is moving with a quantum mechanical velocity proportional to the momentum $\vec{p}$. The principal reason for imposing such a limitation is that the model that will be described in the next chapter is a 1+1 dimensional one, though one can obtain some information on it in more spatial dimensions. Structures in $3+1$ dimensions of the Poincaré generators as

well as other types of operators affected by the latter limitations have been completely determined in the case of spherically symmetric extended objects[56] as well as in the case of non-spherically symmetric objects[39] in 2+1 dimensions.

In order to calculate the generators of the Poincaré transformations in 1+1 dimensions, let us write the algebra these generators must satisfy:

$$[P,H] = 0 \ , \tag{1.57a}$$

$$[P,K] = iH \ , \tag{1.57b}$$

and

$$[H,K] = iP \ . \tag{1.57c}$$

P is the spatial translation generator (total momentum), H is the time translation generator (Hamiltonian) and K is the generator of the Lorentz boost. This algebra together with the asymptotic condition and ($\hbar = 1$),

$$[q,P] = i \tag{1.58}$$

with

$$P = p \ , \tag{1.59}$$

will determine uniquely all the generators. The quantum coordinate q in (1.58) is a Schrödinger-type of operator and is time-independent.

The operator algebra (1.57) together with equation (1.58) determines the position operator up to a unitary transformation. One can therefore always choose it as follows,

$$q = -\frac{1}{2}\{K, H^{-1}\} \ . \tag{1.60}$$

This position operator is just the Newton-Wigner position operator[24,37] which determines, in this case, the free motion of the center of mass of the extended object in configuration space.

The boost generator is then determined as follows,

$$K = -\frac{1}{2}\{q,H\} \quad . \tag{1.61}$$

Now since P and H commute, the asymptotic Hamiltonian is then independent of the position operator and using (1.59) one obtains,

$$H = H(p,n_k) \tag{1.62}$$

where n_k is the number operator ($n_k = \alpha_k^\dagger \alpha_k$).

One can also derive,

$$[q,H] = i\,\frac{dH}{dp} = ipH^{-1} \quad . \tag{1.63}$$

The latter equation together with relation (1.62) determines H to be,

$$H = [p^2 + M^2(n_k)]^{\frac{1}{2}} \ . \tag{1.64}$$

Equation (1.64) contains a mass operator $M(n_k)$ which can be determined from the asymptotic condition. This condition for the Heisenberg field $\psi^f(x)$ can be written as,

$$\psi^f(\vec{x},t) - \langle 0_F|\psi^f(\vec{x},t)|0_F\rangle \xrightarrow[\substack{t\to\pm\infty \\ |\vec{x}-\vec{q}|\to\infty}]{} \psi_o(\vec{x},t) \tag{1.65}$$

where

$$\Lambda(\partial)\psi_o(\vec{x},t) = 0 \ . \tag{1.66}$$

The spatial limit in (1.65) is the same as equation (1.53)

and the time limit is the one defined through the LSZ formalism. For a state where the root mean square value of p is negligibly small, the energy of the system given by the matrix elements of the total Hamiltonian is then weakly equal to the matrix elements of the following asymptotic Hamiltonian,

$$H(n_k) = \int d^3k\ \omega_k\ n_k + M_o \tag{1.67}$$

where ω_k is the energy of the free quanta and M_o is the mass of the extended object. This is so since (1.67) must be free according to (1.65) and (1.66). For this particular state ($|p \to 0\rangle$) one then obtains the following Lorentz invariant,

$$H^2(n_k) - P^2\Big|_{p\to 0} = M^2(n_k)\ . \tag{1.68}$$

Equation (1.68) determines the mass operator,

$$M(n_k) = H_o + M_o \tag{1.69}$$

where H_o is the same as in equation (1.46).

The Heisenberg representation of the quantum coordinate is now given by,

$$Q(t) = e^{iHt}\ q\ e^{-iHt}\ . \tag{1.70}$$

From (1.63) and (1.70) one obtains,

$$i[H,Q(t)] = \dot{Q} = pH^{-1} \tag{1.71}$$

and

$$\ddot{Q} = 0\ . \tag{1.72}$$

Therefore,

$$Q(t) = q + \frac{p}{H} t . \tag{1.73}$$

This shows that the extended object, although classically static, is moving with the quantum velocity $\dot{Q} = pH^{-1}$.

Using the latter asymptotic forms for the Poincaré generators, it is then possible to obtain important information concerning the dynamical map of the Heisenberg field $\psi^f(x)$.

Since the Poincaré transformations are transformations of c-number space-time coordinates $(\vec{x},t)$ we then expect that they will be induced by Q-number transformations of the physical operators that will appear in the dynamical map. It is then necessary for them to show up only through specific combinations with the c-number coordinates. This requirement leads to the existence of the generalized coordinates $(\vec{X},T)$ which are functions of the c-number space-time coordinates and the physical operators. The generalized coordinates can be uniquely determined through the condition of c-Q transmutation with respect to coordinate transformations together with the requirement that they become the c-number coordinates $(\vec{x},t)$ when quantum mechanical operators are disregarded. In 1+1 dimensions the c-Q transmutation condition applied to the generalized coordinates can be expressed as,

$$X(x',t';S) = X(x,t;S(\theta)) \tag{1.74a}$$

and

$$T(x',t';S) = T(x,t;S(\theta)) , \tag{1.74b}$$

where S stands for any physical operator and θ is a transformation parameter. The primes on the left-hand side denote the c-number transformed coordinates. The right-hand side is defined as,

$$X(x,t;S(\theta)) \equiv e^{i\Omega\theta} \; X(x,t;S) \; e^{-i\Omega\theta} \tag{1.75a}$$

and

$$T(x,t;S(\theta)) \equiv e^{i\Omega\theta} \; T(x,t;S) \; e^{-i\Omega\theta} \quad , \tag{1.75b}$$

where Ω is the generator of the transformation.

In 1+1 dimensions the generalized coordinates take the following forms,

$$X = \left(1 + \frac{p^2}{M(H+M)}\right) [x - Q(t)] \tag{1.76a}$$

and

$$T = \frac{M}{H} t - \frac{p}{H} [x - Q(t)] \quad , \tag{1.76b}$$

where M is the mass operator and H is the asymptotic Hamiltonian.

One can then write the dynamical map of $\psi^f(x,t)$ as,

$$\psi^f(x,t) = \tilde{\psi}^f(X,T) \quad . \tag{1.77}$$

Equations (1.76a) and (1.76b) lead to,

$$\partial^2 \equiv \partial_o^2 - \partial_1^2 = D_o^2 - D_1^2 \equiv D^2 \quad , \tag{1.78a}$$

where

$$D^1 \equiv \frac{\partial}{\partial X} \qquad \text{and} \qquad D^o \equiv \frac{\partial}{\partial T} \quad . \tag{1.78b}$$

Now since the differential operator $\Lambda(\partial)$ is the Klein-Gordon operator,

$$\Lambda(\partial) = -\partial^2 - \mu^2 \quad , \tag{1.79}$$

because $\psi^f(x,t)$ is a scalar field, one can then re-write the Heisenberg equation (1.3) as,

$$(-D^2 - \mu^2)\tilde{\psi}^f(X,T) = F[\tilde{\psi}^f(X,T)] \ , \tag{1.80}$$

where (1.77) and (1.78) have been used. It satisfies all the transformation properties of the similar theory without an extended object. Obviously the scalar field must transform as,

$$\psi^f(x',t') = \tilde{\psi}^f(X(\theta),T(\theta)) \ , \tag{1.81}$$

where,

$$X(\theta) \equiv X(x,t;S(\theta)) \tag{1.82a}$$

and

$$T(\theta) \equiv T(x,t;S(\theta)) \quad . \tag{1.82b}$$

II. THE INTERACTING MESON-FERMION-SOLITON MODEL

A Physical Survey of the Model

In this section I intend to describe the Quantum Field Theory of an interacting scalar meson field with a spinor field in 1+1 dimensions. The theoretical interest of such a model resides in the fact that when there is a soliton in the theory, it interacts with both types of quanta and creates a much richer physical situation. For instance, Fermion fields can be trapped by the soliton and then develop zero-energy modes. New quantum operators will be seen to appear and will create a much richer representation for the Hilbert space. Another interesting aspect will consist of studying the transformation properties of the spinor field when the extended object is present. The spinor field operator will be seen to split into two distinct parts which take care separately of coordinates and spinor transformations under Poincaré transformations. It will then be possible to construct an operator S_α from the part taking care of spinor transformations together with the quantum operator associated with the Fermion zero-energy mode. A supersymmetry algebra can then be constructed from the operator S_α together with the usual Poincaré generators. Therefore Bosons are transformed into Fermions and Fermions into Bosons at the level of the physical fields in the soliton sector.

The model is also interesting from the viewpoint of experimental Physics since it has application in quasi one-dimensional systems in Condensed Matter Physics. For example the polyacetylene molecule has a linear chain structure in which electrons and phonons can interact. When there is spontaneous symmetry breaking of the chain structure, a soliton can be created by condensation of optical phonons and the local phenomenon of charge fractionalization arises.

In this model the soliton will be assumed static and topological.

In what will follow the metric tensor is chosen as,

$$g_{oo} = - g_{11} = 1 \tag{2.1}$$

and the Dirac matrices have the following representation,

$$\gamma_o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad . \tag{2.2}$$

The Lagrangian density of our model is written as,

$$\mathcal{L} = \mathcal{L}_o + \mathcal{L}_I \tag{2.3a}$$

where,

$$\mathcal{L}_o = \frac{1}{2}\, \partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\, \mu^2 \phi^2 + \bar{\psi}[i\gamma_\mu \partial^\mu - m]\psi \tag{2.3b}$$

and

$$\mathcal{L}_I = - V(\phi) - \bar{\psi}\psi U(\phi) \quad . \tag{2.3c}$$

Here ϕ is a one-component hermitian scalar field and ψ is a two-component spinor field. We also have $\bar{\psi} \equiv \psi^\dagger \gamma_o$. Equation (2.3b) is the free part of the Lagrangian (2.3a), and (2.3c)

describes the interaction. One must mention that $V(\phi)$ and $U(\phi)$ do not contain derivatives of the field ϕ.

The Euler-Lagrange equations lead to the following field equations,

$$(-\partial^2 - \mu^2)\phi = \frac{\delta V(\phi)}{\delta\phi} + \frac{\delta U(\phi)}{\delta\phi}\bar{\psi}\psi \tag{2.4}$$

and

$$(i\not{\partial} - m)\psi = U(\phi)\psi \quad . \tag{2.5}$$

One sees from (2.4) and (2.5) that the spinor fields interact with the scalar field as well as with the soliton through $U(\phi)$.

From the considerations developed in the previous chapter, one knows that the total momentum P of the system calculated from the Lagrangian (2.3) must be weakly equal to the conjugate momentum p of the quantum coordinate. Therefore the only change required in the set of Poincaré generators obtained in Chapter I is the inclusion of the annihilation and creation operators of the Fermion fields through the number operator in the mass operator $M(n_k)$. Following (1.77) to (1.80), one can re-write equations (2.4) and (2.5) as,

$$(-D^2 - \mu^2)\tilde{\phi}(X,T) = V_1[\tilde{\phi}(X,T)] + U_1[\tilde{\phi}(X,T)]\bar{\psi}(x,t)\psi(x,t) \tag{2.6}$$

and

$$(i\not{\partial} - m)\psi(x,t) = U[\tilde{\phi}(X,T)]\psi(x,t) \quad , \tag{2.7}$$

where

$$U_1(\phi) \equiv \frac{\delta U}{\delta\phi} \qquad \text{and} \qquad V_1(\phi) \equiv \frac{\delta V}{\delta\phi} \; . \tag{2.8}$$

Since the Poincaré generators must induce the coordinate transformations in the spinor fields as well, this implies that the c-number coordinates in $\psi(x,t)$ must appear through the generalized coordinates (1.76a) and (1.76b). However, since $\psi(x,t)$ must transform as a spinor under Lorentz transformations, the structure of its dynamical map written in terms of the physical operators will not be as simple as in the case of the scalar field $\phi(x,t)$.

Although it is possible to discover much of the structure of the spinor fields through a perturbative analysis of the field equations, there is a very simple way of determining the exact structure by examining the transformation properties of the field equations when Lorentz invariants are expressed in terms of the physical operators. Equations (1.78a) and (1.78b) show that there exists a set of matrix operators (Γ_o, Γ_1) such that,

$$\gamma_\mu \partial^\mu = \Gamma_\mu D^\mu \ , \tag{2.9}$$

where D^μ has been defined in (1.78b). Using the explicit forms (1.76a) and (1.76b) for the generalized coordinates, one finds the unique realization,

$$\Gamma_o = \frac{1}{M} (\gamma_o H - \gamma_1 p) \tag{2.10a}$$

and

$$\Gamma_1 = \gamma_1 + \frac{p}{M} [\frac{\gamma_1 p}{H + M} - \gamma_o] \ . \tag{2.10b}$$

Again M is the mass operator and H is the asymptotic Hamiltonian. From the latter expressions it is easy to show that,

$$\{\Gamma_\mu,\Gamma_\nu\} = \{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu} \quad . \tag{2.11}$$

This implies the existence of the following similarity transformation,

$$\Gamma_\mu = A\,\gamma_\mu A^{-1} \quad . \tag{2.12}$$

Inserting equation (2.9) into the field equation (2.7) and making use of the above similarity transformation,one obtains,

$$[i\not{D} - m]A^{-1}\psi(x,t) = U[\tilde{\phi}(X,T)]A^{-1}\psi(x,t) \tag{2.13a}$$

where,

$$\not{D} \equiv \gamma_\mu D^\mu \quad . \tag{2.13b}$$

Defining,

$$\tilde{\psi} \equiv A^{-1}\psi(x,t) \quad , \tag{2.14}$$

equation (2.13a) tells us that $\tilde{\psi}$ is a function of the generalized coordinates. So,

$$\tilde{\psi} = \tilde{\psi}(X,T) \quad . \tag{2.15}$$

Therefore,

$$\psi(x,t) = A\tilde{\psi}(X,T) \quad ,$$

and equation (2.13a) becomes,

$$[i\not{D} - m]\tilde{\psi}(X,T) = U[\tilde{\phi}(X,T)]\tilde{\psi}(X,T) \quad . \tag{2.17}$$

Since $\tilde{\psi}(X,T)$ clearly transforms as a scalar under Lorentz transformations, the matrix A must take care of the spinor part of the transformations. The structure of the spinor field is then exactly determined. In order to find a unique

operator A, one needs to restrict the transformation (2.12) further. The supplementary condition can be found by inserting equation (2.16) into the field equation (2.6),

$$(-D^2 - \mu^2)\tilde{\phi}(X,T) = V_1[\tilde{\phi}(X,T)] + U_1[\tilde{\phi}(X,T)]\tilde{\psi}^\dagger(X,T)A^\dagger\gamma_o A\tilde{\psi}(X,T). \tag{2.18}$$

Requiring that,

$$A^\dagger\gamma_o A = \gamma_o \ , \tag{2.19}$$

equation (2.18) becomes,

$$(-D^2 - \mu^2)\tilde{\phi}(X,T) = V_1[\tilde{\phi}(X,T)] + U_1[\tilde{\phi}(X,T)]\bar{\tilde{\psi}}(X,T)\tilde{\psi}(X,T) \ , \tag{2.20}$$

while A is determined by (2.12) and (2.19). The result is,

$$A = \frac{1}{\sqrt{2M(H+M)}}[H + M + \gamma_o\gamma_1 p] \tag{2.21a}$$

which gives,

$$A^{-1} = \frac{1}{\sqrt{2M(H+M)}}[H + M - \gamma_o\gamma_1 p] \ . \tag{2.21b}$$

It is easy to check that,

$$A = A^\dagger \ . \tag{2.22}$$

Observe at this point that everything so far derived from this model has been studied in one or two spatial dimensions in the case of a non-spherically symmetric extended object and in one, two or three spatial dimensions in the case of a spherically symmetric object. A group theoretical study of this problem has also been done and it has been shown that the matrix operator A is the Lorentz boost matrix operator for the field ψ with quantum velocity $\vec{p}\,H^{-1}$.

It is then possible to generalize the result to fields with arbitrary spin[33].

To show now that zero-energy Fermion modes can appear in this theory requires the solution of the field equations (2.4) and (2.5) with the help of perturbation theory[46]. The method consists of the introduction of a power counting parameter λ into the Lagrangian, the derivation of the corresponding field equations and then the expansion of the solutions as power series of the parameter λ. The field equations can then be solved iteratively and the condition for the emergence of the Fermion zero-energy mode will be clearly seen.

Let us then make the following substitution in the Lagrangian (2.3a),

$$\mathcal{L}(\phi,\psi) \longrightarrow \lambda^{-2}\, \mathcal{L}(\lambda\phi,\lambda\psi) \quad . \tag{2.23}$$

The limit $\lambda \to 1$ should be performed at the end of the computation.

The field equations (2.4) and (2.5) become

$$(-\partial^2 - \mu^2)\phi = \lambda^{-1} V_1(\lambda\phi) - \lambda U_1(\lambda\phi)\bar{\psi}\psi \tag{2.24}$$

and

$$(i\partial\!\!\!/ - m)\psi = U(\lambda\phi)\psi \quad . \tag{2.25}$$

We can expand the quantum fields as,

$$\psi = \sum_{m=0}^{\infty} \lambda^m \psi_m \tag{2.26}$$

and

$$\phi = \sum_{m=-1}^{\infty} \lambda^m \phi_m \quad . \tag{2.27}$$

The term with $n = -1$ in equation (2.27) describes the c-number soliton field of the theory.

As in Chapter I, one can expand the interacting terms $V(\lambda\phi)$ and $U(\lambda\phi)$ as a functional Taylor series about the soliton solution. For the ℓ-th order of this Taylor series one defines,

$$V_\ell \equiv \frac{\delta^\ell}{\delta\phi_{-1}^\ell} V(\phi_{-1}) \tag{2.28a}$$

and

$$U_\ell \equiv \frac{\delta^\ell}{\delta\phi_{-1}^\ell} U(\phi_{-1}) \ . \tag{2.28b}$$

Each term in this Taylor series consists of normal products of the scalar field $\phi(x)$. When equation (2.27) is inserted into the Taylor expansion for $V(\lambda\phi)$ and $U(\lambda\phi)$, one ignores the non-commutativity among different terms in (2.27). We are then working in the tree approximation. In order for the theory to be renormalizable, the interacting part of the Lagrangian density must contain the counterterms. They contribute in each order of the expansions (2.26) and (2.27) and are introduced in each order of the Taylor series for the interacting terms $U(\lambda\phi)$ and $V(\lambda\phi)$ as follows,

$$V_\ell = \sum_{m=0}^{\infty}{}' \lambda^m V_\ell^m \tag{2.29a}$$

and

$$U_\ell = \sum_{m=0}^{\infty}{}' \lambda^m U_\ell^m \tag{2.29b}$$

where the prime means that the summation is restricted to even values of m. This is so since the quantum corrections

arise from a contraction of a pair of fields. The cases $m=0$ and $m=2$ are called the tree and the one-loop approximation respectively. Inserting (2.29a) and (2.29b) into the field equations (2.24) and (2.25) and equating powers of λ one obtains,

$$(-\partial^2 - \mu^2)\phi_n = \sum \frac{1}{\ell !} [V^m_{\ell+1} \phi_{\alpha_1} \ldots \phi_{\alpha_\ell} + U^m_{\ell+1} \phi_{\beta_1} \ldots \phi_{\beta_\ell} \bar{\psi}_r \psi_s] \tag{2.30}$$

and

$$(i\not{\partial} - m)\psi_n = \sum \frac{1}{\ell !} U^m_\ell \phi_{\alpha_1} \ldots \phi_{\alpha_\ell} \psi_r \ . \tag{2.31}$$

The summation in (2.30) is restricted by,

$$\ell + m + \alpha_1 + \ldots + \alpha_\ell = n \ , \quad \ell + m + \beta_1 + \ldots + \beta_\ell + r + s = n + 2 \tag{2.32a}$$

and restricted by,

$$\ell + m + \alpha_1 + \ldots + \alpha_\ell + r = n \tag{2.32b}$$

in equation (2.31).

For the cases $n=-1$ and $n=0$, one has,

$$(-\partial^2 - \mu^2)\ \phi_{-1} = V^o_1 \ , \tag{2.33}$$

$$(-\partial^2 - \mu^2)\ \phi_o = V^o_2\ \phi_o \ , \tag{2.34}$$

and

$$(i\not{\partial} - m)\ \psi_o = U^o_o\ \psi_o \ . \tag{2.35}$$

Equation (2.33) is the Euler equation for the soliton and equation (2.34) is the stability equation with the self-consistent potential V^o_2. Those cases have been studied in Chapter I and lead to quantum soliton states. Equation (2.35) is similar to (2.34) since it represents a

Fermion field trapped by the c-number time independent potential U_o^o created by the soliton. Since ψ_o is the linear term in the dynamical map (2.26) of ψ, it can be chosen to realize an irreducible representation of the Hilbert space of the theory. As in the Boson field case, there can be scattered as well as bound state solutions to (2.35). The general solution representing the particle-like excitations has the form,

$$\Psi_o(x,t) = \sum_i [\eta_i(x) e^{-i\omega_i t} a_i + \xi_i(x) e^{i\omega_i t} b_i^\dagger] + \int dk [\eta(x,k) e^{-i\omega_k t} a(k) + \xi(x,k)\, e^{i\omega_k t} b^\dagger(k)] . \quad (2.36)$$

Here $\eta_i(x)$ and $\eta(x,k)$ are negative frequency bound state and scattered wave functions while $\xi_i(x)$ and $\xi(x,k)$ are the corresponding ones with positive frequency. The set $\{a_i, a(k), b_i^\dagger, b^\dagger(k)\}$ is the set of creation and annihilation operators which obey anticommutation relations.

It is now easy to show that the following function satisfies equation (2.35) with zero energy (time-independent solution),

$$v(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(x) \quad (2.37a)$$

where,

$$u(x) = C \exp[- mx - \int_0^x dy\ U_o^o(y)] \quad . \quad (2.37b)$$

The normalization constant C is finite if,

$$\lim_{x \to \pm\infty} (m + U_o^o) \gtrless 0 \quad . \quad (2.38)$$

The complete solution to (2.35) can then be written as,

$$\psi_o(x,t) = au(x) + \Psi_o(x,t) \tag{2.39}$$

where,

$$a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \alpha \quad . \tag{2.40}$$

Here α is the new operator associated with the zero-energy mode (2.37). The problem of how it realizes an irreducible representation of the Hilbert space will be treated in the second section of this chapter and will be shown to have to do with symmetries associated with the spinor fields. The specific spinor form (2.40) (Majorana spinor) depends upon the choice (2.2) for the representation of the Dirac matrices. In general one can write the operator a as $s\alpha$ where s is a two-components Majorana spinor ($s = Cs^{+T}$ with C being the charge conjugation matrix). Now since the operator α realizes an irreducible representation of the Hilbert space, it anticommutes with the spinor annihilation and creation operators while commuting with the corresponding operators for the Boson field as well as with the quantum mechanical operators (q,p). The function (2.37) together with the wave functions for the particle-like excitations form a complete orthonormal set.

Now the following equal time anticommutation relations (E.T.A.C.R.) for the physical spinor field,

$$\{\psi_{o_\alpha}(x,t_x), \psi^\dagger_{o_\beta}(y,t_y)\}\delta(t_x - t_y) = \delta_{\alpha\beta}\ \delta^{(2)}(x-y) \tag{2.41a}$$

and

$$\{\psi_{o_\beta}(x,t_x),\psi_{o_\beta}(y,t_y)\}\delta(t_x-t_y) = \{\psi^\dagger_{o_\alpha}(x,t_x),\psi^\dagger_{o_\beta}(y,t_y)\}\delta(t_x-t_y)$$
$$= 0 \quad , \tag{2.41b}$$

imply the existence of the operator $\alpha^\dagger$ which satisfies together with α,

$$\{\alpha,\alpha^\dagger\} = 1 \tag{2.42a}$$

and

$$\{\alpha,\alpha\} = \{\alpha^\dagger,\alpha^\dagger\} = 0 \ . \tag{2.42b}$$

As a final remark in this section, let us show that it is possible to construct an operator S_α which together with the Poincaré generators, realizes a supersymmetry algebra.

From equations (1.61) and (2.21a), one has,

$$i[K,A] = -\frac{1}{4}\,[\gamma_o,\gamma_1]\,A \quad . \tag{2.43}$$

The latter relation checks that the matrix A takes care of the spinor part of the Lorentz transformations of the spinor fields $\psi(x)$. Furthermore since the zero-energy mode of ψ_o has zero energy, the operator a does not appear in the asymptotic Hamiltonian. Therefore,

$$[a,H] = [a,K] = 0 \ . \tag{2.44}$$

Defining,

$$S \equiv \sqrt{2M}\;A\,\gamma_o\,a \tag{2.45a}$$

and

$$\bar{S} \equiv S^\dagger\gamma_o = \sqrt{2M}\;a^\dagger\gamma_o A\,\gamma_o \ , \tag{2.45b}$$

and using the Poincaré algebra (1.57) together with (2.43), (2.44) and (2.45), one then shows that the following algebra is realized,

$$[P^{\mu},S_{\alpha}] = 0 \ , \tag{2.46a}$$

$$i[K,S_{\alpha}] = -\frac{1}{4}\ [\gamma_o,\gamma_1]_{\alpha\beta}\ S_{\beta} \quad , \tag{2.46b}$$

and

$$\{S_{\alpha},\bar{S}_{\beta}\} = (\gamma_{\mu})_{\alpha\beta}\ P^{\mu} + i(\gamma_o\gamma_1)_{\alpha\beta}\ M \ . \tag{2.46c}$$

Here $P^{\mu} \equiv (H,P) = (H,p)$. The above algebra is a supersymmetry algebra with central charge M which is the mass operator. It is a symmetry of the theory at the level of the physical fields.

The Quantum Numbers of the Soliton

This section is devoted to the construction of an irreducible representation of the Hilbert space for the theory described by the Lagrangian (2.3). In the special case where the spinor fields do not carry internal symmetry indices, the phase symmetry is the only continuous symmetry associated with the latter fields and leads to a conserved quantity, the Fermion number of the theory. Since it is a good quantum number, the representation for the Hilbert space must contain eigenstates of the Fermion number operator. A study of the charge conjugation properties of the Fermion number operator as well as of the operator S_{α} defined in (2.45a) leads to the fact that the vacuum in the soliton sector possesses eigenstates of the Fermion number operator with eigenvalues $\pm 1/2$ when the spin degree of freedom is ignored. Therefore there is no neutral state in the soliton sector. This result differs from the one

obtained when there is no soliton in the theory. This phenomenon is called the charge fractionalization mechanism.

Besides the above algebraic method, there exists another calculation scheme that enables us to discover the representation for the Hilbert space in the soliton sector[26]. It consists of the solution of the dynamical map of the spinor fields and the explicit construction of the operators leading to the conserved quantities. Since we are interested in the no-particle state in the soliton sector, only the physical operator corresponding to the zero-energy Fermion mode will be relevant. This method can be easily extended to cases where the spinor fields carry internal symmetry indices. In such cases new conserved quantities will appear and the multiplicity of the soliton state will increase. When the Lagrangian is invariant under the transformation group SU(n) ⊗ U(1) in the fundamental representation with respect to the generators of SU(n), it will be shown that a vacuum state in the soliton sector with all quantum numbers equal to zero does not exist.

For generality, we extend the model to,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 - V(\phi) + \mathrm{Tr}\,\bar{\psi}[i\partial\!\!\!/ - m - U(\phi)]\psi, \qquad (2.47)$$

where the trace is taken over internal symmetry indices.

The Heisenberg equation (2.5) is then replaced by,

$$[i\partial\!\!\!/ - m - U(\phi)]\,\psi^c = 0\,, \qquad (2.48)$$

where c is the internal symmetry index.

The existence of the latter symmetry index does not change the perturbative structure of the solution for the Fermion field since the scalar field does not have internal degrees of freedom. Therefore when condition (2.38) is satisfied, the Fermion field ψ^c can develop a zero-energy mode. However since it carries an internal symmetry index, so does the operator α associated with the latter mode. By correspondence with equation (2.39) and from the above considerations, the dynamical map of ψ^c is then written as,

$$\psi^c = s\, u\alpha_c + \Psi_o^c + \ldots \quad . \tag{2.49}$$

Our task is to study how the physical operators $(\alpha_c, \alpha_c^\dagger)$ act on the Hilbert space. Since they carry an internal symmetry index, the anticommutation relations (2.42a) and (2.42b) must be generalized to,

$$\{\alpha_c, \alpha_d^\dagger\} = \delta_{cd} \tag{2.50a}$$

and

$$\{\alpha_c, \alpha_d\} = \{\alpha_c^\dagger, \alpha_d^\dagger\} = 0 \ . \tag{2.50b}$$

The latter relations are obtained from the E.T.A.C.R. (2.41a) and (2.41b) generalized to our case.

The Lagrangian density (2.47) is invariant under the following global symmetry group transformations,

$$\psi \rightarrow \exp[-i\vec{L}\cdot\vec{\theta}]\psi \tag{2.51a}$$

and

$$\psi \rightarrow \exp[-i\theta_o]\psi \ . \tag{2.51b}$$

Here L^i are the classical generators of the non-abelian semi-simple Lie group SU(n) in the fundamental representation, with n^2-1 parameters θ_i. They satisfy the algebra,

$$[L^i, L^j] = C^{ij}{}_k L^k \tag{2.52}$$

with structure coefficients $C^{ij}{}_k$. Equation (2.51b) has only one parameter, θ_o. This is the residual abelian phase symmetry U(1) of the Lagrangian.

An explicit construction of Noether currents lead to the following conserved quantities,

$$F_i = \frac{(L^i)_{cd}}{2} \int dx\, [\psi^{c\dagger}, \psi^d] \tag{2.53a}$$

and

$$Q = \frac{1}{2} \int dx\; \mathrm{Tr}[\psi^\dagger, \psi] \;. \tag{2.53b}$$

Q is the Fermion number operator constructed from the requirement of U(1) invariance while F_i is constructed from invariance under SU(n) transformations. The commutators correspond to the normal ordering prescription and insure that the latter operators have vanishing vacuum expectation value when there is no soliton in the theory.

Making use of the dynamical map (2.49), we re-write equations (2.53a) and (2.53b) as,

$$F_i = \frac{(L^i)_{cd}}{2} [\alpha_c^\dagger, \alpha_d] + \text{physical particles terms} \tag{2.54a}$$

and

$$Q = \frac{1}{2} \mathrm{Tr}[\alpha^\dagger, \alpha] + \text{physical particles terms.} \tag{2.54b}$$

Now the Hilbert space of the theory is constructed as the direct product of the extended Fock space with the

irreducible representation $|S\rangle$ for the physical operators $(\alpha_c, \alpha_c^\dagger)$. The vacuum in the soliton sector can be written as,

$$|0\rangle = |0_F\rangle \otimes |p\rangle \otimes |S\rangle \quad . \tag{2.55}$$

In order to find the representation $|S\rangle$, we must study the effects of the operators (2.54a) and (2.54b) on $|0\rangle$ with $|p\to 0\rangle$. In such a case only the first term in (2.54a) and (2.54b) will have a non-vanishing contribution. Since the classical generators L^i of SU(n) are traceless making use of the algebra (2.50a) and (2.50b) leads to, (2.54a) and (2.54b) becoming,

$$F_i = (L^i)_{cd}\alpha_c^\dagger \alpha_d \tag{2.56a}$$

and

$$Q = \sum_{c=1}^{n} \alpha_c^\dagger \alpha_c - \frac{n}{2} \quad . \tag{2.56b}$$

An important feature of the operator α_c is that it is nilpotent,

$$\alpha_c \alpha_c = \frac{1}{2}\,[\alpha_c, \alpha_c] + \frac{1}{2}\{\alpha_c, \alpha_c\} = 0 \quad . \tag{2.57}$$

The state $|S_c\rangle$ is therefore two-dimensional: $|\uparrow\rangle$, $|\downarrow\rangle$. Note that there could be also a trivial one-dimensional representation for $|S_c\rangle$. However the presence of a central charge in the supersymmetry algebra (2.46) prohibits such a representation.

Define,

$$|\uparrow\rangle \equiv |S_c = 1\rangle \tag{2.58a}$$

and

$$|\downarrow\rangle \equiv |S_c = 0\rangle \quad . \tag{2.58b}$$

Since there are n operators α_c in the fundamental representation of SU(n), one defines their action on $|S\rangle$ as,

$$\alpha_c|S\rangle \equiv \alpha_c|S_1 \ldots S_c \ldots S_n\rangle = (-1)^{\sum_{b<c} S_b} \delta_{1,S_c}|S_1 \ldots \bar{S}_c \ldots S_n\rangle \tag{2.59a}$$

and

$$\alpha_c^\dagger|S\rangle \equiv \alpha_c^\dagger|S_1 \ldots S_c \ldots S_n\rangle = (-1)^{\sum_{b<c} S_b} \delta_{0,S_c}|S_1 \ldots \bar{S}_c \ldots S_n\rangle . \tag{2.59b}$$

Here $|\bar{S}_c\rangle$ is the flipped state of $|S_c\rangle$ and the multiplicity of $|S\rangle$ is 2^n.

The action of the number operator $\alpha_c^\dagger\alpha_c$ is then easily found to be,

$$\alpha_c^\dagger\alpha_c|S_1 \ldots S_c \ldots S_n\rangle = \delta_{1,S_c}|S_1 \ldots S_c \ldots S_n\rangle . \tag{2.60}$$

Now the diagonal operators among the set of operators (2.56a) and (2.56b) will give rise to a set of eigenvalues from which the irreducible representation $|S\rangle$ is determined. From the knowledge of the general form for the n-1 diagonal classical generators L^i_D of SU(n), one can calculate explicitly the eigenvalues of $|S\rangle$. The general form for L^i_D is easily found to be,

$$L^i_D = \frac{1}{\sqrt{2i(i+1)}} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 1 & & & & \\ & & & -i & & & \\ & & & & 0 & & \\ & 0 & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \quad (i=1,\ldots,n-1) \tag{2.61}$$

with i 1's and n-i-1 0's on the diagonal. The normalization factor has been determined from the normalization

condition for the generators L^i of SU(n),

$$Tr(L^iL^j) = \frac{1}{2}\,\delta_{ij} \quad . \tag{2.62}$$

Inserting (2.61) into the expression (2.56a) for F_i and re-naming the diagonal operators as E_i, one obtains,

$$E_i = \frac{1}{\sqrt{2i(i+1)}}\left(\sum_{c=1}^{i} \alpha_c^{\dagger}\alpha_c - i\alpha_{i+1}^{\dagger}\alpha_{i+1}\right) \quad (i=1,\ldots,n-1). \tag{2.63}$$

Making use of (2.56b), (2.60) and (2.63), one finds,

$$E_i|S_1\ldots S_n\rangle = \frac{1}{\sqrt{2i(i+1)}}\left(\sum_{c=1}^{i} \delta_{1,S_c} - i\delta_{1,S_{i+1}}\right)|S_1\ldots S_n\rangle \tag{2.64a}$$

and,

$$Q|S_1\ldots S_n\rangle = \frac{1}{2}\sum_{c=1}^{n} (\delta_{1,S_c} - \delta_{0,S_c})|S_1\ldots S_n\rangle \quad . \tag{2.64b}$$

The latter operators completely determine the representation $|S\rangle$ and one does not need to construct the Casimir operators explicitly. Table I lists the quantum numbers of the soliton up to the case where the theory is invariant under SU(3) $\otimes$ U(1). The eigenvalues of E_i are denoted by e_i and the ones for the Fermion number operator Q are denoted by q. We then have the following representation,

$$|S\rangle = |q;\, e_1\ldots e_{n-1}\rangle \quad . \tag{2.65}$$

When the Fermion number of a state is zero, the eigenvalue of the operator E_{n-1}, never vanishes. This leads to the general result that the soliton always carries at least one non-zero quantum number.

Table I. Quantum number of soliton vacua.

U(1)

	q
$\|1\rangle$	1/2
$\|0\rangle$	-1/2

SU(2) ⊗ U(1)

	q	e_1
$\|11\rangle$	1	0
$\|01\rangle$	0	-1/2
$\|10\rangle$	0	1/2
$\|00\rangle$	-1	0

SU(3) ⊗ U(1)

	q	e_1	e_2
$\|111\rangle$	3/2	0	0
$\|011\rangle$	1/2	-1/2	$-1/2\sqrt{3}$
$\|101\rangle$	1/2	1/2	$-1/2\sqrt{3}$
$\|001\rangle$	-1/2	0	$-1/\sqrt{3}$
$\|110\rangle$	1/2	0	$1/\sqrt{3}$
$\|010\rangle$	-1/2	-1/2	$1/2\sqrt{3}$
$\|100\rangle$	-1/2	1/2	$1/2\sqrt{3}$
$\|000\rangle$	-3/2	0	0

When the only symmetry group is U(1), which leads back to the original model, the soliton is seen to be an energy degenerate doublet with Fermion number $\pm 1/2$. This is called the charge fractionalization mechanism. Higher symmetry leads to an increase of the multiplicity for the soliton states. A pure soliton state with all quantum numbers set equal to zero does not exist.

It is important to realize that the equation for the spinor fields is of the Dirac type, that is, a first order differential equation in both time and space derivatives. Because of this property, it was possible to obtain the important normalizability condition (2.38) insuring the existence of the zero-energy mode of the spinor field.

Finally let us note that models in two[22] and three[17,21] space dimensions have been studied. The vortex-Fermion system[22] is a particular example in two spatial dimensions where Fermion zero-energy modes can appear and lead to new quantum numbers. Here again the requirement of a Dirac-type equation is important if such modes are to exist.

The Charge Fractionalization Mechanism in Condensed Matter Physics

The phenomenon of charge fractionalization has been discussed by field theoreticians involved in High Energy Physics[17,21,22,46] as well as in Condensed Matter Physics[1,4,18,47,48,49]. However, it is in Condensed Matter Physics that field theoreticians have looked for

realistic examples where models with fractionally charged solitons explain experimental data[1].

The purpose of this section is therefore to show qualitatively and succinctly how the phenomenon of charge fractionalization emerges in quasi one-dimensional charge-density-wave (CDW) systems like the polyacetylene molecule and the niobium triselenide molecule ($NbSe_3$).

A one-dimensional CDW system[10,40] is a linear molecular chain where the optical phonon fields associated with harmonic oscillations of the lattice formed by the nuclei interact with the conduction electrons in such a way that a spatially periodic CDW is created throughout the lattice. It can be shown that such a regime arises with the spontaneous symmetry breaking of the reflexion symmetry of the phonon field. Since this phenomenon appears in the mean field approximation where the lattice displacements are treated as c-numbers (condensation of the phonon field in the vacuum), CDW's are seen to arise from the interaction of the conduction electrons with the extended object created by the condensation of the phonon field. CDW's are therefore ground state excitations. In addition to this, an electronic energy gap is created and the conduction electrons become effectively massive.

In the case of the dimerized trans-polyacetylene molecule, the conduction band is half filled with only one electron per site and the wavelength of the CDW is twice the lattice spacing. In the case of the niobium triselenide,

the conduction band is a quarter filled with two paired electrons at each four sites. The wavelength of the corresponding CDW is four lattice spacings.

Let us now see in more detail how the charge fractionalization mechanism occurs in the polyacetylene molecule. The case of the niobium triselenide molecule is interesting from the experimental viewpoint and will be briefly outlined after the following discussion on the polyacetylene molecule.

1. The Polyacetylene Molecule

The polyacetylene molecule has a linear chain structure as shown in figures 1, 2 and 3. Figures 1 and 2 show the two possible configurations of the dimerized trans-polyacetylene without defect in the chain structure. The existence of two such phases implies a double degeneracy of the ground state. Figure 3 shows the coexistence of the two ground states in the same chain. The structure defect interpolating the two phases is the soliton.

Su et al.[47,48,49] have developed recently a discrete field theoretical model to explain stable structure defects in the polyacetylene molecule. The model is one-dimensional implying the neglect of transversal interactions between chains. The conduction electron in this model hops only from site to site along a single chain. Beside the treatment of electrons, oscillations of the chain structure are included through the configuration coordinate u_n. The

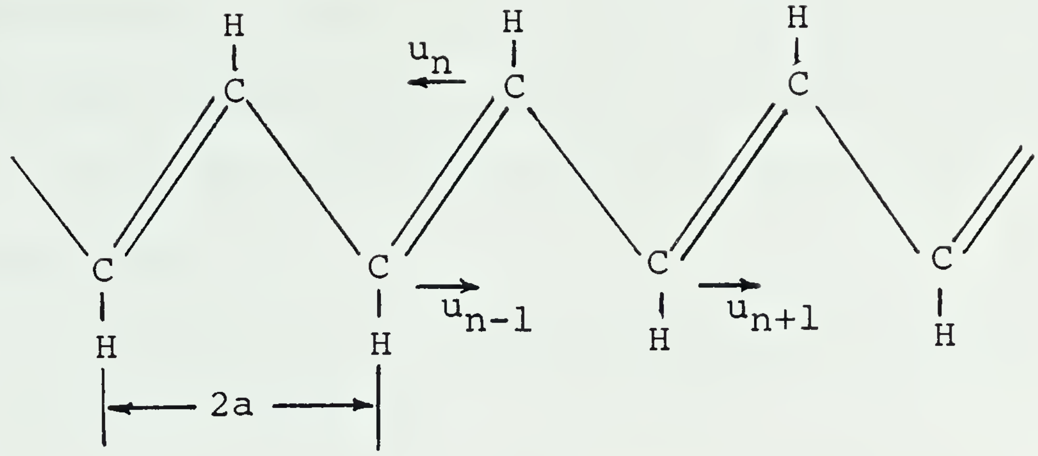

Fig. 1. A-phase of the trans-polyacetylene molecule.

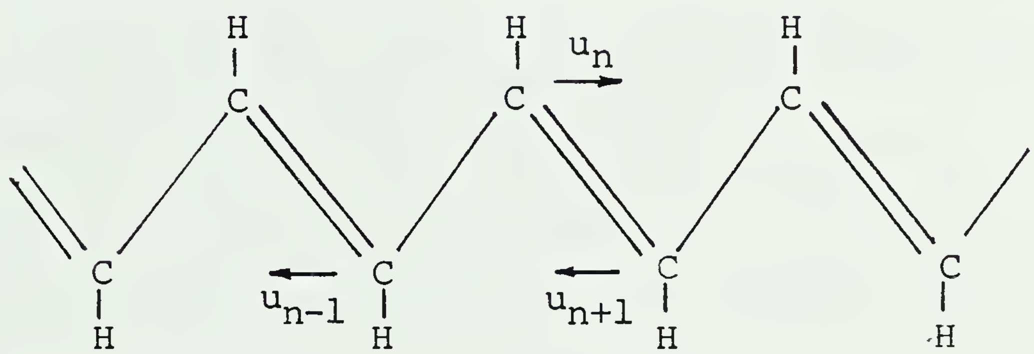

Fig. 2. B-phase of the trans-polyacetylene molecule.

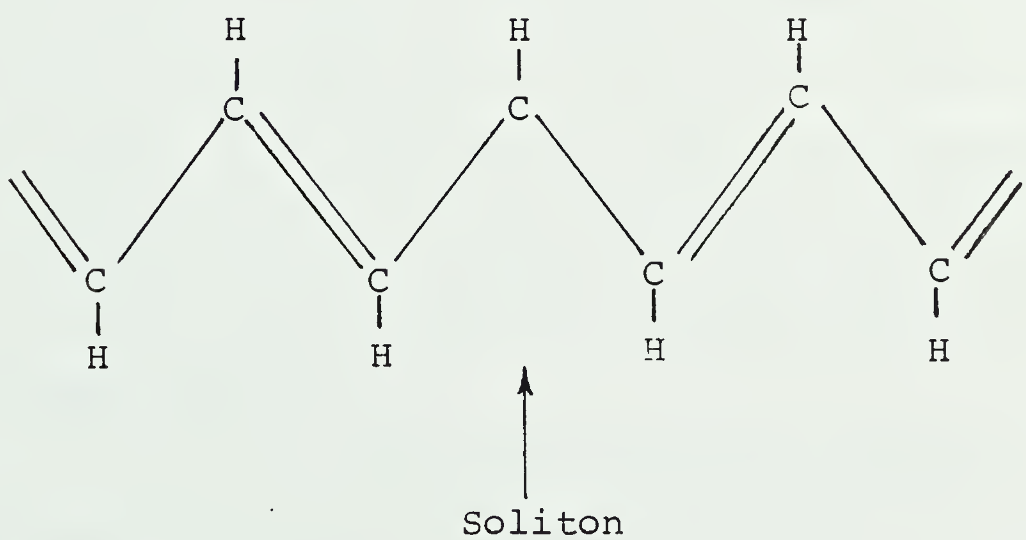

Fig. 3. Soliton in the chain structure of trans-polyacetylene.

latter coordinate obeys a canonical commutation relation with its conjugate momentum $M\dot{u}_n$ (M being the mass of the group CH) and therefore describes quantum phonon fields in the continuum limit.

The Hamiltonian of the Su-Schrieffer-Heeger (SSH) model is then written as,

$$H = -t_o \sum_{n,s} (C^\dagger_{n+1,s} C_{n,s} + C^\dagger_{n,s} C_{n+1,s}) + \frac{1}{2} \sum_n M\dot{u}_n^2 + K(u_{n+1} - u_n)^2$$
$$+ \alpha \sum_{n,s} (u_{n+1} - u_n)(C^\dagger_{n+1,s} C_{n,s} + C^\dagger_{n,s} C_{n+1,s}) \quad . \qquad (2.66)$$

Here the set $\{C_{ns}, C^\dagger_{ns}\}$ is the set of annihilation and creation operators of the electron at site n. The index s denotes the spin and takes the value $\pm 1/2$ while t_o is the energy needed for an electron to hop between two neighboring sites.

The first two terms in (2.66) constitute the free part of the Hamiltonian while the last term is a linear electron-phonon interaction with coupling constant α. Note that the free phonon part of the Hamiltonian is the sum of harmonic oscillator motions of the group CH over all sites n. When there is no interaction, the ground state of the theory is stable and is the phonon vacuum together with all negative energy Fermi electron sea states doubly filled up to the Fermi energy E_F.

Up to now no constraint has been imposed on the configuration coordinate and equation (2.66) does not describe the real physical situation in which single and double

bonds alternate in the chain. One must then impose the following condition,

$$u_n = (-1)^n u \quad . \tag{2.67}$$

This condition insures us that the displacement changes sign alternatively from site to site, which is consistent with the picture of long single bond and short double bond (u_n always points towards the double bond direction).

Inserting condition (2.67) into the Hamiltonian (2.66) and diagonalizing the resulting Hamiltonian gives,

$$H = \frac{1}{2} \sum_n M\dot{u}_n^2 + \sum_{ks} E_k (n_{ks}^c - n_{ks}^v) + 2NKu^2 \quad . \tag{2.68}$$

Here n_{ks}^c and n_{ks}^v are the conduction and valence electron number operators respectively and N is the total number of CH groups in the chain. The second sum in (2.68) runs over the two spin states of the electron and the first Brillouin zone of the lattice ($-\pi/2a \leq k \leq \pi/2a$ with lattice separation a). Note that the electron energy E_k is shown to have a gap. The energy of the ground state is obtained by treating u as a c-number (the soliton) and putting $n_{ks}^c = 0$ and $n_{ks}^v = 1$ into (2.68).

A minimization of the ground state energy with respect to the classical displacement u shows that the vacuum is degenerate and has two stable symmetric minima $\pm u_o$. This is a spontaneous symmetry breaking of the reflexion symmetry (equivalent to the translation symmetry). These two vacua correspond to the chain structure illustrated in figures 1 and 2. However when a soliton is created,

both vacua can coexist as shown in figure 3. The soliton is then the boundary domain between the two stable ground states and interpolates between the corresponding order parameters. This soliton is static and possesses an anti-soliton with opposite sign. We can take them as infinitely separated in the lattice. In this limit they do not interact with each other and become topological.

Now in order to relate the above discrete model to our previous considerations in earlier sections, one must note that the continuous limit ($a \to 0$) of the SSH model (known as the Takayama-Lin-Liu-Maki (TLM) model[50]) leads back to a theory of interacting scalar and fermion fields similar to the one described in earlier sections[23]. The optical phonon is associated to the massive meson field of (2.3) while the conduction electron with energy gap becomes the corresponding massive spinor field. The optical phonon creates then a classical extended object (soliton) by condensation in the vacuum.

In the continuous limit, both models being similar, one can therefore expect the appearance of zero-energy electron modes in the SSH model leading to the charge fractionalization mechanism as in our previous Lagrangian model (2.3). Since the zero-energy modes emerge as a result of the electron-soliton interaction, they will be localized on the soliton which then acquires spin and charge quantum numbers. The relations among quantum numbers of the soliton have been discussed in the second section of this chapter.

However, an argument based on the completeness of the electronic wave functions has been developed and gives some physical insight with respect to the appearance of fractional quantum numbers. Assuming the existence of the electron zero-energy mode, the argument is as follows.

The completeness on the electronic eigenstates without soliton is written as,

$$\int_{-\infty}^{\infty} dE\ \rho_{nn}(E) = 1 \ , \tag{2.69}$$

where,

$$\rho_{nn}(E) \equiv \sum_{s} |\psi_s(n)|^2\ \delta(E - E_s) \ . \tag{2.70}$$

$\rho_{nn}(E)$ is the electronic density at site n. Now since this density is a symmetric function of the energy E, one has,

$$1 = \int_{-\infty}^{\infty} dE\ \rho_{nn}(E) = 2 \int_{-\infty}^{0} dE\ \rho_{nn}(E) \ . \tag{2.71}$$

When the soliton is created however, one must add the contribution of the zero-energy mode to the latter completeness condition. This can be written as,

$$2 \int_{-\infty}^{0} dE\ \rho'_{nn}(E) + |\phi_o(n)|^2 = 1 \ , \tag{2.72}$$

where $\rho'_{nn}(E)$ is the electronic density when the soliton is present. This leads directly to,

$$\int_{-\infty}^{0} dE\ \Delta\rho_{nn}(E) = \frac{1}{2}\ |\phi_o(n)|^2 \ , \tag{2.73}$$

where,

$$\Delta\rho_{nn}(E) \equiv \rho_{nn}(E) - \rho'_{nn}(E) \ . \qquad (2.74)$$

Equation (2.73) tells us that half a state per spin is missing from the infinite set of negative energy Fermi sea states in the case where there is a soliton. When the spin of the electron is neglected, the soliton is then seen to carry Fermion number $\mp 1/2$ depending on whether or not it is occupied. When the spin is taken into consideration, one complete electron state is missing. The unoccupied state has Fermion number equal to +1 and no spin since all spins are paired. Adding one electron to this state makes it neutral while obtaining the two spin configurations of the electron. We then have Fermion number equal to zero with two spin states $\pm 1/2$. The latter state can also be filled with a second electron having a different spin configuration. This creates a soliton state with Fermion number -1 and total spin zero.

Finally it should be emphasized that here too the requirement of a Dirac-type equation (first order differential equation) for the conduction electrons (in the continuous limit) must be imposed. This is so since in the above considerations, the electron zero energy mode has been assumed to exist by analogy with the Lagrangian model (2.3). This requirement is not always fulfilled in Solid State Physics of two or three dimensional space as in the case of the superconductivity model[53].

2. The Niobium Triselenide ($NbSe_3$) Molecule

Let us turn briefly now to the case of the niobium triselenide ($NbSe_3$)[1]. It is a CDW system with period of four lattice spacings. As mentioned previously the conduction band is a quarter filled with two paired electrons at each four sites. A CDW with no defect is said to be commensurate with the lattice spacing. Figure 4 reproduces schematically the electron distribution on the lattice.

Since one period is four lattice units, the system can show up in four configurations (phases) having the same energy. The ground state is therefore four times degenerate. The removal of two electronic states from the conduction band at one site of the chain will create a phase defect (the soliton). Since the ground state was four times degenerate it takes four solitons in the lattice to recuperate the loss in phase (figure 5). From charge conservation, the phase defects must carry electronic states. Since two electronic charges 2e have been transferred from the conduction band to the electronic states of four solitons, each of them must carry fractional charge e/2. Note that CDW with phase defects are called incommensurate CDW's while the phase defect itself has been called a discommensuration[36]. The latter correspond therefore to the localized zero-energy electronic modes discussed earlier.

From the experimental viewpoint,the above molecule is interesting since,when one applies an electric field on the system, the charged phase defect can move across the

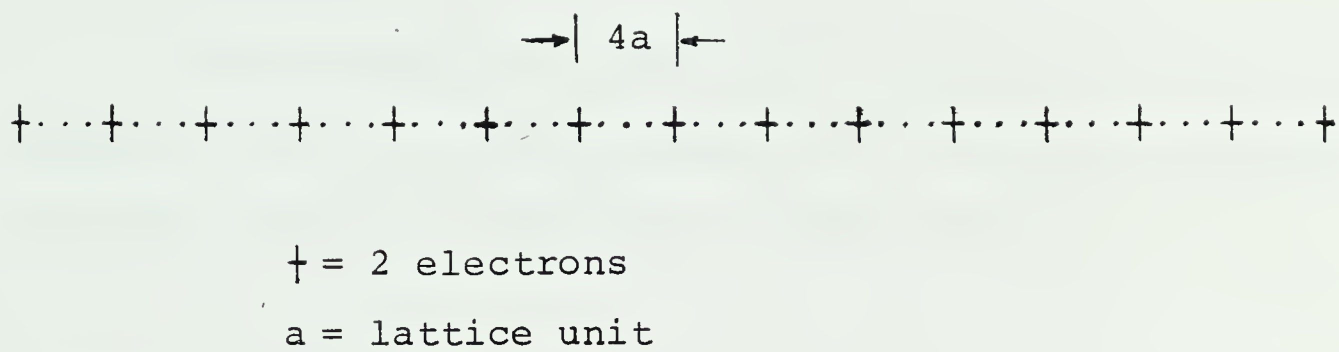

Fig. 4. Schematic structure of $NbSe_3$ without soliton.

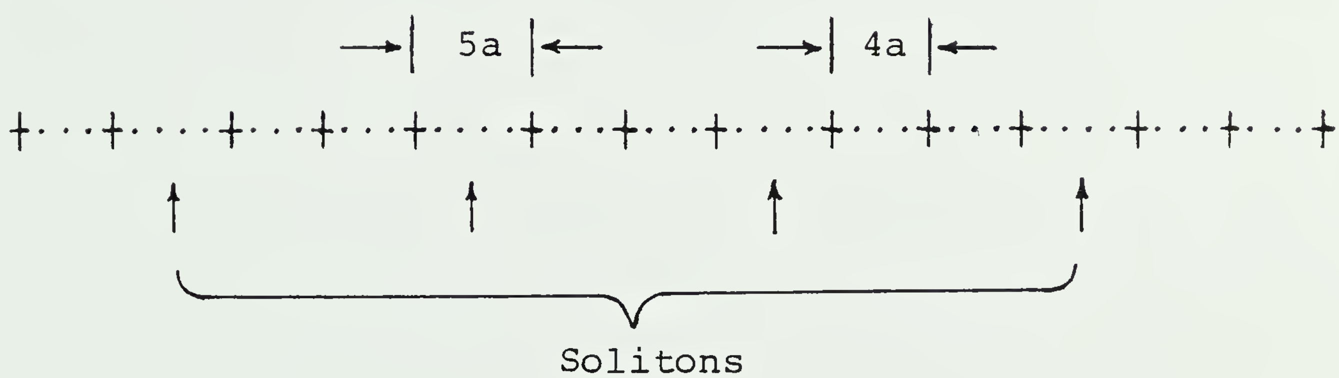

Fig. 5. Schematic structure of $NbSe_3$ with solitons.

lattice leading to the possibility of current measurements. Such measurements have been performed and lead to charge values ranging between 0,3 e and 0,4 e. One experiment even yields 0,47 e. It is argued that the discrepancies between these numbers and e/2 may be due to experimental error and/or real effects[1].

Nevertheless one can take the latter values as positive evidence of the existence of fractionally charged extended objects in Condensed Matter Physics.

CONCLUSION

In the course of the present work, Quantum Field Theory with extended objects has been reviewed through the treatment of a self-interacting scalar field model and an interacting scalar-spinor-soliton model in 1+1 dimensions.

The results can be classified as follows.

1) In both models the Hilbert space of the theory differs from the one without an extended object. As a result of the emergence of a new type of interaction (the quanta-soliton interaction), the Fock space has been enlarged to include scattered and bound states of quanta with the quantum mechanical soliton.

2) Particularly important among the bound states are the so-called zero-energy modes with which one constructs an irreducible representation for the pure soliton state of the theory.

3) An important consequence of the existence of extended objects is the effect on the asymptotic condition in Quantum Field Theory. Usually defined through the LSZ formalism, the asymptotic condition requires that as $t \to \pm\infty$ the Heisenberg field weakly becomes the in-field (free field) from which the Hilbert space is constructed. When an extended object is created however, the latter time limit only separates quanta from interacting among themselves while still feeling the effects from the extended object through the self-consistent potential. A spatial limit then

separates quanta from the object and we are led to the fully free case.

4) Because of the above reformulation of the asymptotic condition, the asymptotic forms for the Poincaré generators differ from the ones in the case when there is no extended object.

5) Using the new asymptotic forms for the Poincaré generators and making extensive use of the c-Q transmutation condition one can find much information on the structure of the dynamical map of Heisenberg fields from their transformation properties. This completely determines the appearance of quantum mechanical operators in the dynamical map. These operators always appear in unique combinations with the c-number space-time coordinates. These combinations are called the generalized coordinates.

6) Particularly interesting is the case of spinor fields which split into two parts, one part taking care of coordinate transformations and which transforms as a Lorentz scalar, and a second part which transforms as a spinor under Lorentz transformations. The latter spinor operator consists of p and H and gives an example of the construction of spinor operators from the tensor operators $\vec{p}$,H (spinorization).

7) Under certain circumstances, the Fermion fields can develop a zero-energy mode through its interaction with the extended object. This mode can carry internal degrees of freedom when the theory is invariant under an internal

symmetry group transformation.

8) Finally, in the task of constructing an irreducible representation for the Hilbert space of the theory, one finds new quantum numbers carried by the soliton. For instance, the charge fractionalization mechanism is a particular case where the soliton carries fractional Fermion number $\pm 1/2$ when the spin degree of freedom is ignored. In the case of the model described in chapter II, this leads to observable physical effects as in the case of the polyacetylene molecule or the niobium triselenide molecule. In general the vacuum in the soliton sector is never "neutral".

REFERENCES

1. BAK, Per, Phys. Rev. Lett., 48, 10 (1982) 692.

2. BJORKEN, J.D. and S.D. DRELL, "Relativistic Quantum Fields", McGraw-Hill Book Co., New York (1965).

3. CALLAN, C. and D. GROSS, Nucl. Phys. B93 (1975) 29.

4. CAMPBELL, D.K. and A.R. BISHOP, Nucl. Phys. B200 [FS4], 2 (1982) 297.

5. CHRIST, N. and T.D. LEE, Phys. Rev. D12 (1975) 1606.

6. COLEMAN, S., Erice Summer School Lecture (ed. A. Zichichi, Plenum Publ. Corp. 1975).

7. DASHEN, R., B. HASSLACHER and A. NEVEU, Phys. Rev. D10 (1974a) 4114.

8. DASHEN, R., B. HASSLACHER and A. NEVEU, Phys. Rev. D10 (1974b) 4130.

9. FADDEEV, L.D. and V.E. KOREPIN, Phys. Reports C42 (1978) 1.

10. FRÖHLICH, H., Proc. Roy. Soc. Ser., A223 (1954) 296.

11. GERVAIS, J.-L. and A. JEVICKI, Nucl. Phys. B110 (1976) 93.

12. GERVAIS, J.-L., A. JEVICKI and B. SAKITA, Phys. Rev. D12 (1975) 1038.

13. GERVAIS, J.-L. and A. NEVEU, Phys. Reports C23 (1976) 237.

14. GERVAIS, J.-L. and B. SAKITA, Phys. Rev. D11 (1975) 2943.

15. GLAUBER, R.J., Phys. Rev. 131 (1963) 2766.

16. GOLDSTONE, J. and R. JACKIW, Phys. Rev. D11 (1975) 1486.

17. GOLDSTONE, J. and F. WILCZEK, Phys. Rev. Lett. 47, 14 (1981) 986.

18. HO, Tin-Lun, Phys. Rev. Lett., 48, 14 (1982) 946.

19. ITZYKSON, C. and J.-B. ZUBER, "Quantum Field Theory", McGraw-Hill Book Co., New York (1980).

20. JACKIW, R., Rev. Mod. Phys., 49, 3 (1977) 681.

21. JACKIW, R. and C. REBBI, Phys. Rev. D13 (1976) 3398.

22. JACKIW, R. and P. ROSSI, Nucl. Phys. B190 [FS3] (1981) 681.

23. JACKIW, R. and J.R. SCHRIEFFER, Nucl. Phys. B190 [FS3] (1981) 253.

24. JORDAN, T.F., J. Math. Phys., 21 (1980) 2028.

25. KOREPIN, V.E. and L.D. FADDEEV, Teor. Mat. Fiz., 25 (1975) 147 [Theor. Math. Phys. (USSR) 25 (1976) 1038].

26. LEBLANC, Y. and G. SEMENOFF, Phys. Rev. D26 (1982) 938.

27. MATSUMOTO, H., G. OBERLECHNER, M. UMEZAWA and H. UMEZAWA, J. Math. Phys., 20 (1979) 2088.

28. MATSUMOTO, H., N.J. PAPASTAMATIOU, G. SEMENOFF and H. UMEZAWA, Phys. Rev. D24 (1981) 406.

29. MATSUMOTO, H., N.J. PAPASTAMATIOU and H. UMEZAWA, Nucl. Phys. B82 (1974) 45.

30. MATSUMOTO, H., N.J. PAPASTAMATIOU and H. UMEZAWA, Nucl. Phys. B97 (1975) 90.

31. MATSUMOTO, H., N.J. PAPASTAMATIOU, H. UMEZAWA and G. VITIELLO, Nucl. Phys. B97 (1975) 61.

32. MATSUMOTO, H., N.J. PAPASTAMATIOU, M. UMEZAWA and H. UMEZAWA, Phys. Rev. D23 (1981) 1339.

33. MATSUMOTO, H., G. SEMENOFF and H. UMEZAWA, "On the Interaction of Quantum Spinor Fields with Extended Objects", U. of Alberta Preprint (1982).

34. MATSUMOTO, H., G. SEMENOFF, H. UMEZAWA and M. UMEZAWA, J. Math. Phys., 21 (1980) 1761.

35. MATSUMOTO, H., P. SODANO and H. UMEZAWA, Phys. Rev. D19 (1979) 511.

36. McMILLAN, W.L., Phys. Rev. B14, 4 (1976) 1496.

37. NEWTON, T.D. and E.P. WIGNER, Rev. Mod. Phys. 21 (1949) 400.

38. OBERLECHNER, G., M. UMEZAWA and Ch. ZENSES, Lett. Nuovo Cimento, 23 (1978) 641.

39. PAPASTAMATIOU, N.J., H. MATSUMOTO and H. UMEZAWA, "The Rotational Mode of Asymmetric Objects in Quantum Field Theory", U. of Alberta Preprint (1982).

40. PEIERLS, R.E., "Quantum Theory of Solids", Clarendon Press, Oxford (1955).

41. RAJARAMAN, R., Phys. Reports, C21 (1975) 227.

42. RICE, M.J., Physics Letters, 71A, 1 (1979) 152.

43. RICE, M.J., A.R. BISHOP, J.A. KRUMHANSL and S.E. TRULLINGER, Phys. Rev. Lett. 36, 8 (1976) 432.

44. SCHWEBER, S.S., "An Introduction to Relativistic Quantum Fields", Harper and Row, Publishers (1961).

45. SEMENOFF, G., H. MATSUMOTO and H. UMEZAWA, J. Math. Phys., 22 (1981) 2208.

46. SEMENOFF, G., H. MATSUMOTO and H. UMEZAWA, Phys. Rev. D25 (1982) 1054.

47. SU, W.P. and J.R. SCHRIEFFER, Phys. Rev. Lett., 46 (1981) 738.

48. SU, W.P., J.R. SCHRIEFFER and A.J. HEEGER, Phys. Rev. Lett., 42 (1979) 1698.

49. SU, W.P., J.R. SCHRIEFFER and A.J. HEEGER, Phys. Rev., B22 (1980) 2099.

50. TAKAYAMA, H., Y.R. LIN-LIU and K. MAKI, Phys. Rev., B21 (1980) 2388.

51. TOMBOULIS, E., Phys. Rev., D12 (1975) 1678.

52. UMEZAWA, H. and H. MATSUMOTO, Symmetries in Science, 411 (ed. B. Gruber and R.S. Millman, Plenum Publ. Corp., New York, 1980).

53. UMEZAWA, H., H. MATSUMOTO and M. TACHIKI, "Thermo-Field Dynamics and Condensed States", North-Holland Publ. (1982).

54. WADATI, M., H. MATSUMOTO and H. UMEZAWA, Phys. Lett., B73 (1978) 448.

55. WIGNER, E., Annals of Mathematics, 40 (1939) 149.

56. YAMANAKA, H., H. MATSUMOTO and H. UMEZAWA, Phys. Rev., D24 (1981) 2607.